

Background Independent Quantum Mechanics, Gravity, and Physics at Short Distance: Some Insights

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In the present discussion Background Independent framework of Quantum Mechanics and its possible implications in the studies of gravity and Physics at short distance are addressed. The expression of the metric of quantum state space $g_{\mu\nu}$ which is intrinsically a quantized quantity, is identified in terms of Compton wavelength as: $[\langle \partial_\mu \psi | \partial_\nu \psi \rangle - \langle \partial_\mu \psi | \psi \rangle \langle \psi | \partial_\nu \psi \rangle] = \frac{1}{\lambda_C^2} (= \frac{m_0^2 c^2}{\hbar^2})$. The discussion also sheds light on the notion of neighborhood in quantum evolution.

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There is a prevailing feeling that either Quantum Mechanics (QM) or General Relativity (GR) or both should pave way for new geometrical feature in QM [1-4]. And an intensive follow up of this call for the extension of standard geometric quantum mechanics [1-8] would be academically rewarding [1]. Physicists studying gravity have also shown considerable interest in the geometric structures in QM in general and projective Hilbert space in specific [1-4, 9, 10]. However, we feel that there is enough information hidden in the standard geometric QM that is yet to be explored. The present discussion aims to address Background Independent framework of Quantum Mechanics and its possible applications in the studies of gravity and Physics at short distance.

To begin with, we briefly discuss the basic tenets of standard geometric QM [1-4, 6, 7], and the background independent settings in which investigations are going on, to make it relevant to studies of gravity. Pure states are points of an infinite dimensional Kähler manifold on $\mathcal{P}(\mathcal{H})$ the complex projective space of the Hilbert space \mathcal{H} . Equivalently $\mathcal{P}(\mathcal{H})$ is a manifold with an almost complex structure. The probabilistic interpretation lies in the definition of geodesic length on the space of quantum states (events).

The space of quantum mechanics (events) becomes dynamical and that the dynamical geometrical information is described in terms of a non-linear diffeomorphism invariant theory in such a way that the space of quantum events is non-linearly inter-related with the Hamiltonian- the generator of quantum dynamics. The distance on the projective Hilbert space is defined in terms of metric, called the metric of the ray space [1-4, 6-10] or the projective Hilbert space \mathcal{P} , is given by the following expression in Dirac's notation:

$$ds^2 = [\langle d\psi | d\psi \rangle - \langle d\psi | \psi \rangle \langle \psi | d\psi \rangle] \quad (1)$$

This can be an alternative definition of the Fubini-Study (FS) metric, valid for an infinite dimensional \mathcal{H} .

The metric in the ray space being treated by physicists as the background independent and space-time independent structure, can play an important role in the construction of a potential "theory of quantum gravity". The demand of background independence in quantum theory of gravity calls for an extension of standard geometric quantum mechanics [1-4]. It is an important insight which can be springboard for our proposed background independent generalization of standard quantum mechanics. For a generalized coherent state, the FS metric reduces to the metric on the corresponding group manifold [2]. Thus, in the wake of ongoing work in the field of quantum geometric formulation, the work in the present discussion may prove to be very useful. The probabilistic (statistical) interpretation of QM is hidden in the metric properties of $\mathcal{P}(\mathcal{H})$. The unitary time evolution is also in a way related to the metrical structure [1, 2, 6-10] with Schrödinger's equation in the guise of a geodesic equation on $CP(N)$. The time parameter of the evolution equation can be related to the quantum metric *via*:

$$(\Delta E)^2 \equiv \langle \psi | H^2 | \psi \rangle - \langle \psi | H | \psi \rangle^2; \quad (2)$$

with $\hbar ds = \Delta E dt$.

And the Schrödinger equation can be viewed as a geodesic equation on $CP(N) = \frac{U(N+1)}{U(N) \times U(1)}$ as:

$$\frac{du^a}{ds} + \Gamma_{bc}^a u^b u^c = \frac{1}{2\Delta E} \text{Tr}(H F_b^a) u^b. \quad (3)$$

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Here $u^a = \frac{dz^a}{ds}$ where z^a denote the complex coordinates on $CP(N)$, Γ_{bc}^a is the connection obtained from the Fubini-Study metric, and F_{ab} is the canonical curvature 2-form valued in the holonomy gauge group $U(N) \times U(1)$. Here, Hilbert space is $N + 1$ dimensional and the projective Hilbert space has dimensions N .

If the metric of quantum states is defined with the complex coordinates in the quantum state space, known as Fubini-Study metric, it lies on the Kähler manifold or $CP(N)$, which is identified with the quotient set $\frac{U(N+1)}{U(N) \times U(1)}$.

The symmetries described by this quotient set have limitations. However, the most appropriate representation that seems to satisfy the almost complex structure criteria is the Grassmannian. By the correspondence principle, the generalized quantum geometry must locally recover the canonical quantum theory encapsulated in $\mathcal{P}(\mathcal{H})$, also with mutually compatible metric and symplectic structure, allows the framework for the dynamical extension of the canonical quantum theory.

The Grassmannian:

$$Gr(C^{N+1}) = \frac{Diff(C^{N+1})}{Diff(C^{N+1}, C^N \times 0)}, \quad (4)$$

in the limit $N \rightarrow \infty$ satisfies the necessary conditions [10]. The Grassmannian is gauged version of complex projective space, which is the geometric realization of quantum mechanics. The utility of this formalism is that gravity embeds into quantum mechanics with the requirement that the kinematical structure must remain compatible with the generalized dynamical structure under deformation [10]. The quantum symplectic and metric structure, and therefore the almost complex structure, are themselves fully dynamical. Time the evolution parameter in the generalized Schrödinger equation is yet not deemed to be global and is thus transformed in terms of the invariant distance. The basic point as threshold of the BIQM is to notice that the evolution equation (the generalized Schrödinger equation) as a geodesic equation can be derived from an Einstein-like equation with the energy-momentum tensor determined by the holonomic non-abelian field strength F_{ab} of the $Diff(\infty - 1, C) \times Diff(1, C)$ type and the interpretation of the Hamiltonian as a charge. Such an extrapolation is logical, since $CP(N)$ is an Einstein space, and its metric obeys Einstein's equation with a positive cosmological constant given by:

$$R_{ab} - \frac{1}{2}Rg_{ab} - \Lambda g_{ab} = 0. \quad (5)$$

The diffeomorphism invariance of the new phase space suggests the following dynamical scheme for the BIQM as:

$$R_{ab} - \frac{1}{2}Rg_{ab} - \Lambda g_{ab} = T_{ab}. \quad (6)$$

Furthermore,

$$\nabla_a F^{ab} = \frac{1}{2\Delta E} H u^b. \quad (7)$$

The last two equations imply *via* Bianchi identity, a conserved energy-momentum tensor

$$\nabla_a T^{ab} = 0. \quad (8)$$

This taken together with the conserved “current” as

$$j^b = \frac{1}{2\Delta E} H u^b; \quad (9)$$

implies the generalized geodesic Schrödinger equation. Thus equations (8) and (9), being a closed system of equations for the metric and symplectic structure do not depend on the Hamiltonian, which is the case in ordinary QM too. Moreover, the requirement of diffeomorphism invariance places stringent constraints on the quantum geometry. We have to have an almost complex structure for the generalized space of quantum events. This extended framework readily implies that the wave-functions labeling the relevant space are themselves irrelevant. They are as meaningless as coordinates in General Relativity.

The metric of the quantum state space has been identified as background independent (BI) metric structure [1-4]. By appearance itself the invariance of the significance of geometric structure in equation (1) is apparent. The reformulation of the geometric QM in this background independent settings gives us many a new insights. Quantum states being unobservable and also due to $Diff(\infty, C)$ symmetry, make no sense physically, only quantum events do. This is quantum counterpart of the corresponding statement of the meaning of space-time events in GR. Probability is generalized and is given by the notation of diffeomorphism invariant distance in the space of quantum configurations.

As discussed repeatedly, the expression in equation (1) is the metric in the BIQM framework that leads to yet another question: what this invariant and constant quantity stands for? The answer is revealed by the Klein-Gordon evolution. And we emphasize that there are interesting facts associated with this geometry of quantum state space that cannot be ignored. The metric of the quantum state space, which is intrinsically a quantized quantity as $g_{\mu\nu} = [\langle \partial_\mu \psi | \partial_\nu \psi \rangle - \langle \partial_\mu \psi | \psi \rangle \langle \psi | \partial_\nu \psi \rangle]$ had originally been derived from the expression $[\langle \psi | H^2 | \psi \rangle - \langle \psi | H | \psi \rangle^2]$. As already shown [1], if we consider the relativistic evolution of quantum states by Klein-Gordon equation, it reveals the reasons that give rise to the this covariant and invariant quantity as:

$$-\psi^* \nabla^\mu \nabla_\mu \psi = \frac{m_0^2 c^2}{\hbar^2} \psi^* \psi. \quad (10)$$

By using the definition of covariant derivative of the quantum state space in Dirac's notation, left hand side of equation (10) could be rewritten in generalized manner [1] and the expression of the metric of quantum state space is obtained as follow:

$$\frac{1}{2}[(\psi^* \nabla_\mu \nabla_\nu \psi) + (\psi^* \nabla_\mu \nabla_\nu \psi)^*] = -[\langle \partial_\mu \psi | \partial_\nu \psi \rangle - \langle \partial_\mu \psi | \psi \rangle \langle \psi | \partial_\nu \psi \rangle]. \quad (11)$$

From which we find an interesting result:

$$[\langle \partial_\mu \psi | \partial_\nu \psi \rangle - \langle \partial_\mu \psi | \psi \rangle \langle \psi | \partial_\nu \psi \rangle] = \frac{m_0^2 c^2}{\hbar^2}. \quad (12)$$

The quantity in the right hand side is familiar one. It can be defined as square of inverse of the Compton's wavelength as:

$$\left(\frac{m_0 c}{\hbar}\right)^2 = \frac{1}{\lambda_C^2}. \quad (13)$$

Thus one can think of the invariant ds^2 in the ray space evolving as multiple of inverse of the Compton wavelength. As this final expression is valid irrespective of the choice of quantum states, we can draw this inference for all quantum states in generality. At long wavelengths, once we map the configuration space to space-time, we have General Relativity. Turning off dynamics the quantum configuration space recovers the canonical quantum mechanics [10]. An equally important clue one comprehends from equation (13) that gives rise to the question, whether the presence of a quantity such as inverse of distance squared imply the signatures of gravity or a cosmological constant in this geometric structure? This is subject of rigorous investigations.

Interestingly, the Compton wavelength at Planck scale:

$$(\lambda_C)_{PlanckScale} = \frac{\hbar}{m_{Pl} c} = 1.6 \times 10^{-33} cms, \quad (14)$$

is precisely the Planck's length. Thus, the lowest value that the Compton wavelength ceases to be, is the Planck's length only.

We know that cosmological constant is the variance in the vacuum energy about zero mean. The variance ΔE as it appeared in one of the original propositions [6] of the metric of quantum states

$$ds^2 = \frac{(\Delta E)^2}{\hbar^2} dt^2, \quad (15)$$

leads to a natural question: what this uncertainty of energy stands for? Also, the conclusion of equation (13) is obvious if the variance in energy ΔE in equation (15) could assume a typical value $(\Delta E)^2 \sim (m_0 c^2)^2$. If the quantum state under consideration is the state of vacuum then it could be the variance in the vacuum energy as:

$$(\Delta E)^2 = \langle 0 | H^2 | 0 \rangle - \langle 0 | H | 0 \rangle^2. \quad (16)$$

It is interesting to note that there is something physical in the right hand side of equation (15) which appears as a geometrical form in the left hand side of the equation. The invariant ds in the metric structure of quantum states is not distance in the dimensional sense, it is neighborhood in the topological sense. It is the infinitesimally small neighborhood implied by this expression which fills the space. This expression of metric of quantum states as it appeared in one of its original propositions [6] was later generalized in the quantum state space. As suggested by T. W. Kibble [5] in the context of proposed generalization of quantum mechanics that the states that are in a sense

defined near vacuum can be represented by vectors in the tangent space T_ν , and that on T_ν one has all the usual structure of linear quantum mechanics expressed in the local coordinates. However, we need to specify what is meant by "nearness" to the vacuum. At each point on the space-time manifold, the space is locally flat. Locally, the vacuum energy is fixed by the quantum theory in the tangent space, which is also the case in the Matrix theory [10]. Gauging QM generically breaks Super-Symmetry. We do not have globally defined super-charges in space-time in the correspondence limit. This also explains- why there is cosmological constant [10]. The detail study will appear elsewhere.

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